Solution 8

1. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that

$$|Ax| \leq \sqrt{\sum_{i,j} a_{ij}^2}\ |x|.$$

Solution. Let $y = Ax$. We have

$$y_i = \sum_j a_{ij} x_j, \quad i = 1, \cdots, n\ .$$

By Cauchy-Schwarz Inequality,

$$|y_i| \leq \sqrt{\sum_j a_{ij}^2} \sqrt{\sum_j x_j^2}\ .$$

Taking square,

$$y_i^2 \leq \sum_j a_{ij}^2 \sum_j x_j^2\ .$$

Summing over i,

$$\sum_i y_i^2 \leq \sum_{i,j} a_{ij}^2 \sum_j x_j^2\ ,$$

and the result follows by taking root.

Note. This result was used in the proof of Proposition 3.5.

2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

Solution. Let $\Phi(x) = Ix + Ax$ so that $\Psi(x) = Ax$ for $x \in \mathbb{R}^n$. By the previous problem,

$$|\Psi(x_1) - \Psi(x_2)| = |A(x_1 - x_2)| \leq \sqrt{\sum_{i,j} a_{ij}^2}\ |x|\ .$$

Take $\gamma = \sqrt{\sum_{i,j} a_{ij}^2} < 1$. Ψ is a contraction and there is only one root of the equation $\Phi(x) = 0$ in the ball $B_r(0)$. However, since we already know $\Phi(0) = 0$, 0 is the unique root. Now, we claim that $I + A$ is non-singular, for there is some $z \in \mathbb{R}^n$ satisfying $(I + A)z = 0$, we can find a small number α such that $\alpha z \in B_r(0)$. By what we have just shown, $\alpha z = 0$ so $z = 0$, that is, $I + A$ is non-singular and thus invertible.

The sharpness of the condition $\sum a_{ij}^2 < 1$ can be seen from considering the 2×2-matrix A where all $a_{ij} = 0$ except $a_{22} = -1$.

Note. See how linearity plays its role in the proof.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$

is a contraction. This provides a justification for Newton's method in finding roots for an equation.

Solution. $T'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$. Since f is C^2 and $f(x_0) = 0, f'(x_0) \neq 0$, it follows that T is C^1 in a neighborhood of x_0 with $T(x_0) = x_0, T'(x_0) = 0$ and there exists some $\rho > 0$

$$|T'(x)| \leq \frac{1}{2}, \quad x \in [x_0 - \rho, x_0 + \rho].$$

As a result, T is a contraction in $[x_0 - \rho, x_0 + \rho]$. By Contraction Mapping Principle, there is a fixed point for T. From the definition of T, this fixed point is a root for the equation $f(x) = 0$.

4. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n), \ x_1 \in [0, 1] \ .$$
Find
   (a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.
   (b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.
   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

Solution. Let $Tx = \alpha x(1 - x)$. The max of T attains at $1/2$ so the maximal value is $\alpha/4$. Therefore, the range of α is $[0, 4]$ so that T maps $[0, 1]$ to itself. Next, 0 is always a fixed point of T. To get no other, we set $x = \alpha x(1-x)$ and solve for x and get $x = (\alpha-1)/\alpha$. So there is no other fixed point if $\alpha \in [0, 1]$. Finally, it is clear that T becomes a contraction when $\alpha \in [0, 1)$, so the sequence $\{x_n\}$ with $x_0 \in [0, 1]$, $x_n = T^n x_0$, always tends to 0 as $n \to \infty$. Although T is not a contraction when $\alpha = 1$, one can still use elementary mean (that is, $\{x_n\}$ is always decreasing,) to show that 0 is an attracting fixed point.

5. Show that every continuous function from $[0, 1]$ to itself admits a fixed point. Here we don't need it a contraction. Suggestion: Consider the sign of $g(x) = f(x) - x$ at $0, 1$ where f is the given function.

Solution. Let $f \in C[0, 1]$. Clearly, if $f(0) = 0$, then 0 is a fixed point. So assume $f(0) \neq 0$. Here we take $f(0) > 0$. Consider the continuous function $g(x) = f(x) - x$. We have $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 \leq 0$. If equality holds, then $f(1) = 1$, 1 is a fixed point. If inequality holds, that is, $g(1) < 0$, by the mean-value theorem there is some $\xi \in (0, 1)$ such that $g(\xi) = 0$, that is, $f(\xi) - \xi = 0$, so ξ is a fixed point. The case $f(0) < 0$ can be handled similarly.

Note. This example shows that every continuous function from $[0, 1]$ to itself, not only contractions, admits a least one fixed point. (But not necessarily unique.) Similar result holds for all continuous maps on a compact, convex subset in $\mathbb{R}^n$ to itself. It is called Brouwer's fixed point theorem.

6. Let f be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is either positive or negative everywhere. This is 2060 stuff.

Solution. $\Rightarrow$. Let g be the inverse of f. When g is differentiable, we can use the chain rule in the relation $g(f(x)) = x$ to get $g'(f(x))f'(x) = 1$, which implies that $f'(x)$

never vanishes. Since f' is continuous, if $f'(x_0) > 0$ at some x_0, we claim f' is positive everywhere. Suppose $f'(x_1) < 0$ at some x_1, by continuity $f'(x_2) = 0$ at some x_2 between x_0 and x_1, contradiction holds. Hence f' is positive everywhere. Similarly, it is negative everywhere when it is negative at some point.

$\Leftarrow$. Let us assume f' is always positive (the other case can be treated similarly.) Let $x < y$ in $[a, b]$. By the mean value theorem, there is some $z \in (x, y)$ such that $f(y) - f(x) = f'(z)(y - x) > 0$, so f is strictly increasing. According to an old result in 2050, a continuous, strictly increasing function maps $[a, b]$ to the interval $[f(a), f(b)]$ and its inverse g is continuous. Then we can use the Carathedory Criterion in 2060 to show that g is differentiable and, in fact, satisfies $g'(f(x)) = 1/f'(x)$.

7. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin \frac{1}{x}, \quad x \neq 0,$$
and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the inverse function theorem?

Solution. $|f(x) - f(0) - (1/2)x| = |x^2 \sin(1/x)| = O(x^2)$, hence f is differentiable at 0 with $f'(0) = 1/2$. Let $x_k = 1/2k\pi, y_k = 1/(2k\pi + 1)$, then $f'(x_k) = -1/2, f'(y_k) = 3/2$. Then it is clear that f is not injective in $I_k = (y_k, x_k)$. Since any neighborhood of 0 must include contain some I_k, this shows that f it has no local inverse at 0. It does not contradict the inverse function theorem because f' is not continuous at 0.

Note. This problem shows that the C^1-condition is needed in the Inverse Function Theorem.

8. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2,\ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

Solution. Let $u = x - x^2, v = y + xy$. The Jacobian determinant is 1 at $(0, 0)$ so there is an inverse in some open set containing $(0, 0)$. Now we can describe it explicitly as follows. From the first equation we have
$$x = \frac{1 \pm \sqrt{1 - 4u}}{2}.$$
From $u(0, 0) = 0$ we must have
$$x = \frac{1 - \sqrt{1 - 4u}}{2} .$$
Then
$$y = \frac{v}{1 + x} = \frac{2v}{1 - \sqrt{1 - 4u}}.$$
We see that the largest domain in which the inverse exists is $\{(u, v) : u \in (0, 1/4), v \in \mathbb{R}\}$.

9. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?

Solution. Let E be an open set in U. We need to show that $F(E)$ is open. Let $y_0 \in F(E)$ and $x_0 \in E$ satisfy $F(x_0) = y_0$. By the Inverse Function Theorem (applied to $F : E \to$

$\mathbb{R}^n$), there are open sets V (in E) and W containing x_0 and y_0 respectively such that $F(V) = W$. In particular, $W \subset F(E)$. Since W is open and contains y_0, there is some $B_r(y_0) \subset W \subset F(E)$, so $F(E)$ is open.

The inverse may not exist. Consider the map $(r, \theta) \to (r\cos\theta, r\sin\theta)$ in $(r, \theta) \in (0, \infty) \times \mathbb{R}$, whose Jacobian determinant is always nonzero. However, it has no inverse.

10. Consider the function

$$h(x, y) = (x - y^2)(x - 3y^2), \quad (x, y) \in \mathbb{R}^2.$$

Show that the set $\{(x, y) : \ h(x, y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

Solution. The Jacobian matrix of h is singular at $(0, 0)$, hence the Implicit Function Theorem cannot apply. Indeed, $h(x, y) = 0$ means either $x - y^2 = 0$ or $x - 3y^2 = 0$. The solution set of $\{(x, y) : \ h(x, y) = 0\}$ consisting of two different parabolas passing the origin.